Math 4650 - Homework # 1
Properties of the real numbers

Part 1 - Computations

1. Find the values of x that satisfy the following equations and draw a picture of the solution set.

   (a) $|x-1| < 2$

   (b) $|-2+x| \leq 4$

   (c) $0 < |x-2| < 0.01$

   (d) $0 < |x-5| \leq 2$

2. Find $\inf(X)$ and $\sup(X)$ if they exist.

   Don't prove anything, just state the answer. Draw a picture.

   (a) $X = \left\{ 5 + \frac{1}{n} \;\middle|\; n \in \mathbb{N} \right\}$

   (b) $X = \left\{ 1 + \frac{(-1)^n}{n} \;\middle|\; n \in \mathbb{N} \right\}$

   (c) $X = \left\{ \frac{1}{1+x^2} \;\middle|\; x \in \mathbb{R} \right\}$

   (d) $X = \left\{ \frac{x}{1+x} \;\middle|\; x \in \mathbb{R} \text{ with } x > -1 \right\}$

   (e) $X = \left\{ x \in \mathbb{R} \;\middle|\; x^2 + 1 < 3 \right\}$

   (f) $X = \left\{ x \in \mathbb{R} \;\middle|\; x^3 \leq 1 \right\}$

Part 2 - Proofs

3. Let $x \geq 0$ be a real number. Suppose that for each $\epsilon > 0$ we have that $x \leq \epsilon$. Prove that $x = 0$.

4. Suppose that S is a non-empty subset of the real numbers. Suppose that the supremum of S exists. Prove that it is unique.

5. Let S be a non-empty subset of the real numbers. Suppose that b is an upper bound for S and $b \in S$. Prove that b is the supremum of S.

6. Suppose that A and B are non-empty subsets of $\mathbb{R}$ that are both bounded from below and above. Further suppose that $A \subseteq B$.

   (a) Prove that $\inf(B) \leq \inf(A) \leq \sup(A) \leq \sup(B)$.

   (b) Disprove by giving a counterexample:
   If $\sup(A) = \sup(B)$ and $\inf(A) = \inf(B)$ then $A = B$.

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7. Let A and B be non-empty subsets of $\mathbb{R}$. Suppose that the supremum of A and supremum of B exist.

   (a) <u>Prove:</u> If $A \cap B$ is non-empty then $\sup(A \cap B) \leq \min\{\sup(A), \sup(B)\}$

   (b) <u>Disprove:</u> If $A \cap B$ is non-empty then $\sup(A \cap B) = \min\{\sup(A), \sup(B)\}$

   (c) <u>Prove:</u> $\sup(A \cup B) = \max\{\sup(A), \sup(B)\}$

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8. Let a, b, x, and y be real numbers. Prove:

   (a) $|a - b| = |b - a|$

   (b) $|ab| = |a|\,|b|$

   (c) $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$ if $b \neq 0$

   (d) If $a < x < b$ and $a < y < b$, then $|x - y| < b - a$.

   (e) $|\,|a| - |b|\,| \leq |a - b|$
